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Transformation of Monochromatic Waves from Deep to Shallow Water

by

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PREFACE

This report is published to provide coastal engineers with a formulation and a set of nomographs for determining the breaking wave characteristics, such as breaking wave height, depth of breaking, and angle of breaking wave with a straight shoreline, as functions of the deepwater wave characteristics: wave height, wave period, and wave angle. This formulation is necessary to determine the littoral drift transport; however, to obtain such results, a review of nonlinear wave transformation is presented. A "hybrid" wave approach based on linear (or Stokes third order) and cnoidal waves is proposed as the best theory from available experimental data. The work was carried out under the coastal structures program of the Coastal Engineering Research Center (CERC).

This report was prepared by Bernard Le Mehaute, Professor and Chairman, and John D. Wang, Associate Professor, Ocean Engineering, Rosenstiel School of Marine and Atmospheric Science, University of Miami, Miami, Florida, under CERC Contract No. DACW72-79-C-0005.

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Comments on this publication are invited.

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Ted E. Bishop

TED E. BISHOP
Colonel, Corps of Engineers
Commander and Director

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CONVERSION FACTORS, U.S. CUSTOMARY TO METRIC (SI) UNITS OF MEASUREMENT

U.S. customary units of measurement used in this report can be converted to metric (SI) units as follows:

Multiply	by	To obtain
inches	25.4	millimeters
	2.54	centimeters
square inches	6.452	square centimeters
cubic inches	16.39	cubic centimeters
feet	30.48	centimeters
	0.3048	meters
square feet	0.0929	square meters
cubic feet	0.0283	cubic meters
yards	0.9144	meters
square yards	0.836	square meters
cubic yards	0.7646	cubic meters
miles	1.6093	kilometers
square miles	259.0	hectares
knots	1.852	kilometers per hour
acres	0.4047	hectares
foot-pounds	1.3558	newton meters
millibars	1.0197×10^{-3}	kilograms per square centimeter
ounces	28.35	grams
pounds	453.6	grams
	0.4536	kilograms
ton, long	1.0160	metric tons
ton, short	0.9072	metric tons
degrees (angle)	0.01745	radians
Fahrenheit degrees	5/9	Celsius degrees or Kelvins[1]

[1]To obtain Celsius (C) temperature readings from Fahrenheit (F) readings, use formula: C = (5/9) (F -32).

To obtain Kelvin (K) readings, use formula: K = (5/9) (F -32) + 273.15.

SYMBOLS AND DEFINITIONS

A	elliptic function
B	elliptic function
C	wave phase speed
E	complete elliptic integral of the second kind
H	wave height
H_o'	unrefracted deepwater wave height
K	complete elliptic integral of the first kind
K_r	refraction coefficient
K_s	shoaling coefficient
L	wavelength
L_c	cnoidal wavelength parameter
L_1	linear wavelength
Q	littoral transport rate
S	beach slope
T	wave period
U	Ursell parameter = HL^2/d^3
V(u,v)	particle velocity
d	stillwater depth
f_H	cnoidal shoaling function
g	gravity
k	wave number
p	pressure
t	time
x-y	cartesian ordinates
α	angle of wave incidence
η	free-surface elevation
ρ	density
$\phi(x,z,t)$	velocity potential
b,o	subscripts which refer to breaking or deepwater values, respectively

TRANSFORMATION OF MONOCHROMATIC WAVES FROM DEEP TO SHALLOW WATER

by
Bernard Le Mehaute and John D. Wang

I. INTRODUCTION

An understanding of many nearshore phenomena relies on the ability to predict the local wave climatology, given a deepwater wave description. For example, a quantitative description of longshore sediment transport is based on a knowledge of the wave characteristics in the surf zone. This report presents methods for determining the changes in the characteristics of a wave traveling over a variable bottom from deep water to shallow water.

The acute sensitivity of the rate of littoral transport to wave breaking characteristics implies an accurate determination of these characteristics. The problem has numerous facets:

(a) Given a deepwater unidirectional monochromatic wave, what are the breaking wave angle, depth of breaking, breaking wave height, and related quantities?

(b) Given a multidirectional deepwater incident wave spectrum, what is the distribution of breaking wave characteristics and the "equivalent" monochromatic wave used to determine the littoral drift?

(c) How should a synoptic wave climatology be treated in order to determine the rate of littoral drift and related quantities?

Only the first problem is addressed in this report. The relevant literature is reviewed, and a new hybrid wave theory is proposed to determine wave breaking characteristics on a sloped plane beach.

II. NONLINEAR WAVE TRANSFORMATION

1. Nonlinear Wave Shoaling.

It is generally assumed that the wave motion over a gentle slope is the same as that on a horizontal bottom, and that there is no reflection nor wave profile deformation. The wave motion is then determined so that the rate of transmission of energy or energy flux is constant over varying depth.

The average energy flux through a vertical plane of unit width perpendicular to the wave propagation is

$$F_{av} = \frac{\rho}{T} \int_t^{t+T} \int_{-d}^{\eta} \left(gz + \frac{p}{\rho} + \frac{1}{2} V^2\right) u\,dz\,dt \qquad (1)$$

where

ρ = density

t = time

T = wave period

d = water depth

η = free-surface elevation

g = gravity acceleration

p = pressure

$V(u,v)$ = particle velocity

z = vertical ordinate

In the general case, linear or nonlinear, where the flow motion can be expressed by a potential function $\phi(x,z,t)$, the Bernoulli equation yields

$$-\phi_t = gz + \frac{p}{\rho} + \frac{1}{2} V^2 \tag{2}$$

and $u = \phi_x$ so that the energy flux becomes

$$F_{av} = -\frac{\rho}{T} \int_t^{t+T} \int_{-d}^{\eta} \phi_t \phi_x dxdt \tag{3}$$

in which case ϕ can be expressed at any order of approximation, such as given by a Stokesian power series. Even though classical solutions for cnoidal waves are irrotational, the potential function is not expressed but rather the solution for (η,u,v) is given; therefore, the energy flux for cnoidal wave is determined from equation (1) where $(V^2 = u^2 + w^2)$.

The results of all the calculations pertinent to linear wave theory and linear wave shoaling are given in Le Mehaute (1976).

Instead of expressing ϕ at a first order of approximation as in the linear wave theory, ϕ is expressed at a higher order in equation (3), the shoaling coefficient $K_S = H/H_O$ becomes not only a function of d/L or d/L_O but also a function of the deepwater wave steepness, H_O/L_O.

This calculation has been performed at a third order of approximation (Le Mehaute and Webb, 1964), and the fifth order of approximation (Koh and Le Mehaute, 1966) based on the third-order solution and fifth-order solution for a Stokesian wave as developed by Skjelbreia and Hendrickson (1960). The first definition of Stokes for the phase velocity is used; i.e., the average horizontal water particle velocity over a wavelength is zero. The results of such investigation are presented in Figures 1 and 2.

The correction ΔH due to nonlinear effects never exceeds 5 percent and is more commonly of the order of 1 percent. These investigations show that:

(a) The nonlinear shoaling coefficient is initially less than the linear coefficient when $d/L_0 \lesssim 0.4$, then becomes larger toward shallow water until the wave breaks.

(b) The Stokesian power series is not uniformly convergent, i.e., the function of d/L of higher order tends toward infinity when d/L tends to small values. Therefore, the "best" order of approximation is not necessarily the highest order. For relatively deep water $d/L > 0.25$, the fifth order of approximation would be the best insofar as wave height transformation is concerned; for very shallow water $d/L < 0.01$, the linear theory would be best. In the intermediate range the third-order theory would be best, and therefore should be preferred overall because of its range of applicability.

The second definition of Stokes for the phase velocity can also be used; the average momentum over a wavelength is zero by addition of a uniform motion. Yamaguchi and Tsuchiya (1976) indicate that the results yield slightly larger values, at most a 7-percent increase for the shoaling coefficient, than the results obtained by Le Mehaute and Webb (1964).

The principle of conservation of energy flux has also been applied to a cnoidal wave, and like the Stokesian wave the results depend on the order of approximation and the definition of phase velocity. All these investigations on cnoidal waves are based on an energy flux such as expressed by equation (1). Masch (1964) was the first to deal with this subject; however, his wave theory is not consistent, even erroneous, (in the table of functions used by Masch in the shoaling of cnoidal wave, the water depth below MWL should be substituted by h_t, the water depth under trough), and the results are presented in a form which is difficult to use. The relation to deepwater wave and sinusoidal theory is not discussed and no attempt is made to follow the shoaling of a specific wave.

A significant contribution to the shoaling of cnoidal waves is given by Iwagaki (1968). Iwagaki treats the case of an approximate solution of cnoidal wave in which he used the second definition of phase velocity as given by Laitone (1961). The approximation is on the value of the elliptic integral which is replaced by a simple function of empirical coefficients. Iwagaki shows that this simplification actually covers a wide range of cases and allows him to simply investigate the shoaling of what he calls "hyperbolic waves." When the energy flux in deep water (as computed using small-amplitude theory) is equated to the energy flux in shallow water, described by first-order hyperbolic waves, Iwagaki obtains

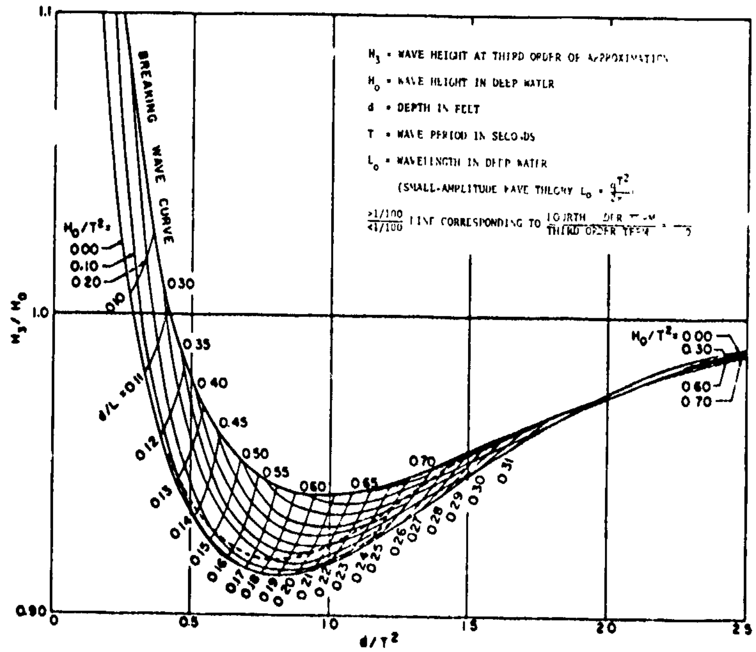


Figure 1. The shoaling coefficient at the third order of approximation (Le Mehaute and Webb, 1964).

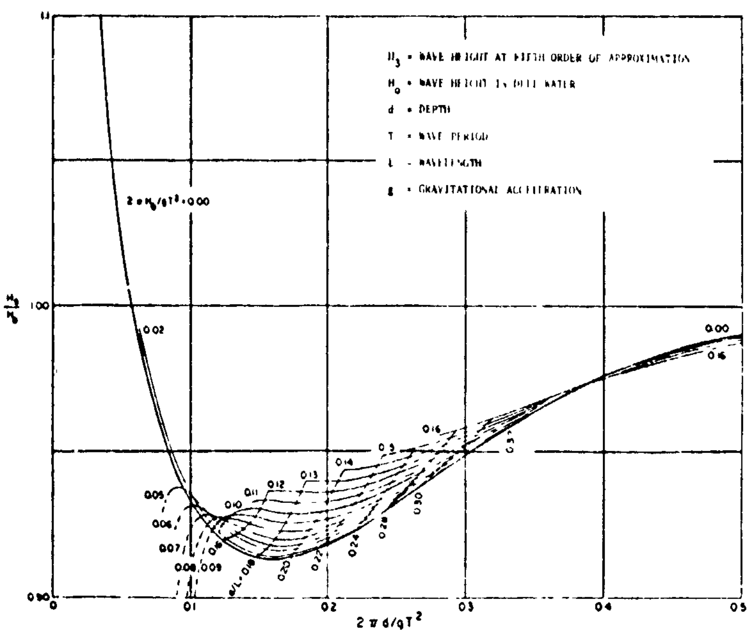


Figure 2. The shoaling coefficient at the fifth order of approximation (Koh and Le Mehaute, 1966).

$$\frac{H}{H_o} = \frac{3}{4}\left(\frac{1}{16}\right)^{2/3}\left(\frac{H_o}{L_o}\right)^{1/3}\left(\frac{d}{L_o}\right)^{-1} \tag{4}$$

According to Iwagaki (1968), this theory yields sufficiently accurate results for Ursell parameter $U \gtrsim 47$. However, as pointed out by Svendsen (1974), the theory of Iwagaki deserves to be regarded as a practical solution to second-order cnoidal waves when the deepwater wave steepness is smaller than 0.02 and the relative water depth is smaller than 0.05. The matching of the Iwagaki hyperbolic wave with the third-order Stokesian wave is shown in Figure 3.

The shoaling of the true cnoidal wave has been investigated by Svendsen and Brink-Kjaer (1972), Svendsen (1974), and Svendsen and Hansen (1977). They also give H/H_n as function of d/L_o and H_o/L_o (Fig. 4) and a computer-printed table. It can then be shown that for large values of Ursell parameters the shoaling coefficient $K_s \to d^{-1}$ instead of $d^{-1/4}$ as given by the Green law (long wave linear theory). Concurrently, Shuto (1974) arrives at very similar results.

Yamaguchi and Tsuchiya (1976) also carry out the same calculation based on the two definitions of the Stokes wave velocity for the cnoidal theory of Laitone (1961) and that of Chappelear (1962). However, an arithmetic error has been found in the Laitone theory (Le Mehaute, 1968).

2. Comparison and Matching Between Various Theories.

As a wave propagates from deep water to shallow water it is theoretically possible to determine the variation of wave height, wavelengths, etc. This could be done by applying the principle of conservation of energy flux to either the linear wave or the nonlinear Stokesian wave, or the cnoidal and solitary wave. Since a Stokesian wave rather applies in deep water, the transformation of water wave should be followed with that theory for the largest value of relative depth d/L_o and then switched to the cnoidal theory when d/L_o becomes small. However, such a scheme implies that the theories can be matched continuously, but there is a priori no reason why the ratio H/H_o should be the same for the value d/L_o which corresponds to the limit of validity of both theories. On the other hand, if the wave heights are matched, then the energy flux will present a discontinuity (Fig. 5). The significant feature is that the cnoidal wave height grows faster with decreasing depth, although at intermediate depth its value is up to 10 percent less than predicted by a Stokesian theory. Waves with wave steepness larger than 2 to 3 percent will break at a depth where the cnoidal wave height is only slightly larger than that of a Stokesian wave. Waves with small wave steepness, however, such as swells, reach much smaller depth before they break and consequently a major part of their shoaling process is governed by the cnoidal wave theory. For these waves, the two theories such as the Stokesian (first order or linear theory) and cnoidal wave at a second order will yield significantly different results.

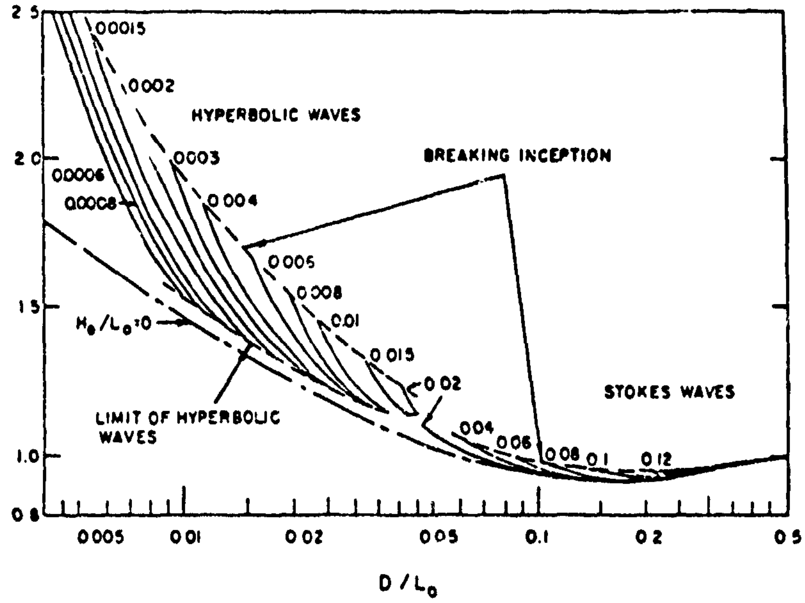


Figure 3. Shoaling coefficient obtained in equating the energy flux between a third-order Stokian wave in deep water and cnoidal (hyperbolic) wave in shallow water (Iwagaki, 1968).

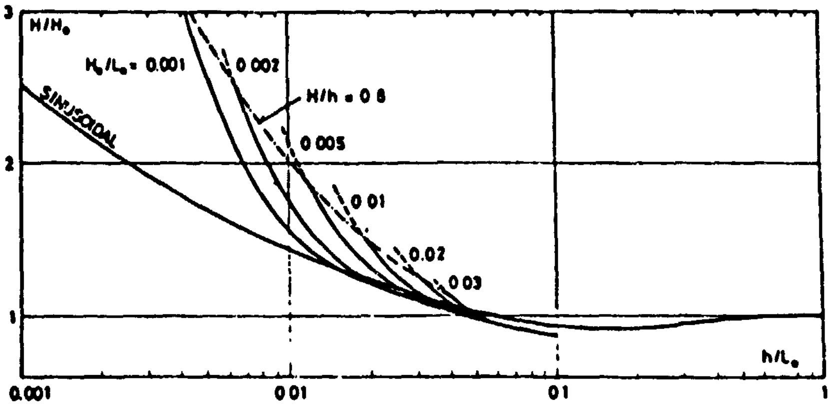


Figure 4. Comparison between shoaling coefficients according to linear and cnoidal wave theories (Svendsen and Brink-Kjaer, 1972).

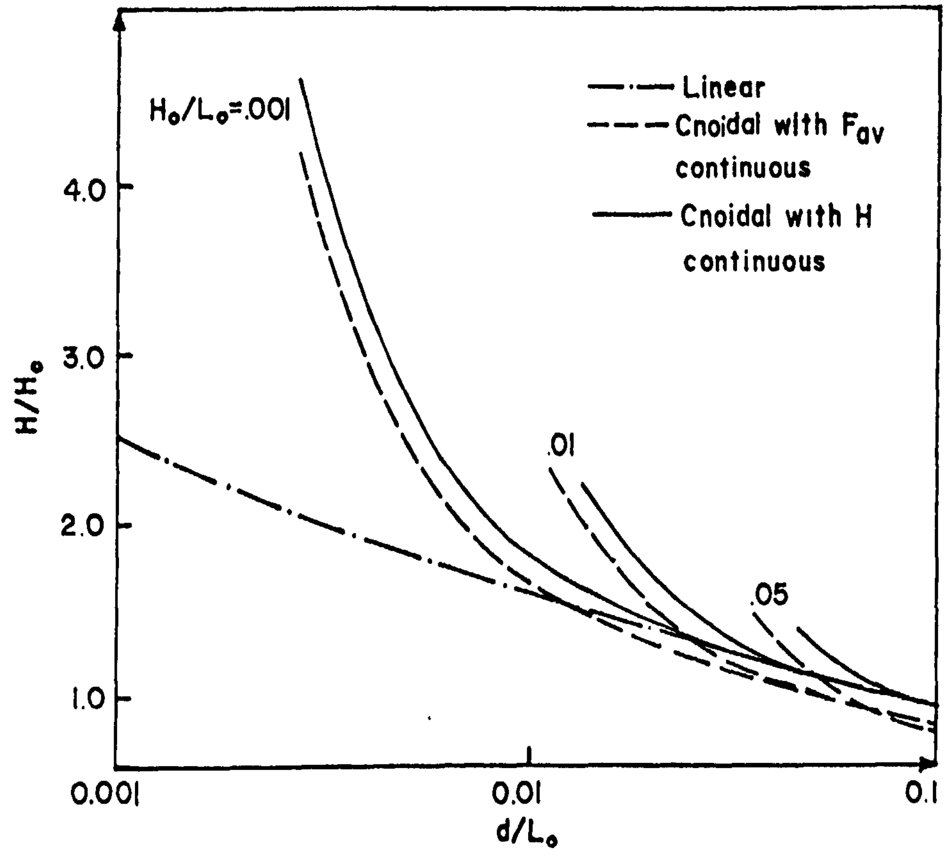


Figure 5. Matching Stokesian (first order) and cnoidal wave theory (Svendsen and Hansen, 1977).

These results (Fig. 5) show that no continuous transition is possible between the two theories. This means that it is not possible to find a value of the water depth, d, where the curves for the two theories fit smoothly together. If the Stokesian theory is used in deeper water and changed to a cnoidal theory when the wave enters shallow water, there will be a discontinuity in the variation of either wave height or wavelength, or both, depending on which water depth is chosen for the switch. Of course, the same will appear for all other quantities such as particle velocities, pressure, etc., and the rate of change of these. Svendsen (1974) shows that the limit of applicability of the cnoidal theory is $d/L_0 < 0.1193$ when H is small. Koh and Le Mehaute (1966) also showed that the limit of applicability of the fifth-order Stokesian wave theory is $d/L_0 > 0.10$ when $H/L_0 = 0.05$ and $d/L_0 > 0.13$ when $H/L_0 = 0.10$ (see Fig. 2).

There is a large difference between Stokesian and cnoidal wave between d/L_0 equal 0.1 and 0.3. In this region no known wave theory fits very well. It could have been expected that a higher order Stokesian theory would be the answer, but the investigation by Koh and Le Mehaute (1966) shows that when d/L_0 decreases the fifth-order approximation represents an even worse approximation than the third order. Similarly, it is found that second-order cnoidal theory is worse than first-order cnoidal theory for large wave steepness. This is inherent to the point that both cnoidal and Stokesian power series expansion in terms of the small parameters h/d and H/L respectively are nonuniformly converging series since the functions of d/L attached to each power term blow up when d/L tends toward small values.

It is interesting that Yamaguchi and Tsuchiya (1976) found that the shoaling coefficient given by Le Mèhaute and Webb (1964) (first definition of Stoke's phase velocity) almost coincides with the shoaling coefficient obtained from cnoidal theory developed by Chappelear (1962) (second definition).

Shuto (1974) attempted to make a synthesis of all these theories in a simple and practical form by empirically matching these solutions. Subsequently, he proposes the following law for practical purposes:

$$0 < \frac{L_o H}{d^2} \leq \frac{30}{2\pi} \quad : \quad \text{The small-amplitude theory applies}$$

$$\frac{30}{2\pi} < \frac{L_o H}{d^2} \leq \frac{50}{2\pi} \quad : \quad \text{Use } Hd^{2/7} = \text{constant}$$

$$\frac{50}{2\pi} < \frac{L_o H}{d^2} \leq \infty \quad : \quad \text{Use } Hd^{5/2}\left[\left(\frac{L_o 2\pi H}{d^2}\right)^{1/2} - 2\sqrt{3}\right] = \text{constant} \qquad (5)$$

These equations seem to be the most realistic to remember from all the theoretical approaches. In the range where both cnoidal and third-order Stokesian theory apply, the values of the shoaling coefficient are very close to each other as shown in Figure 6 (Flick, 1978).

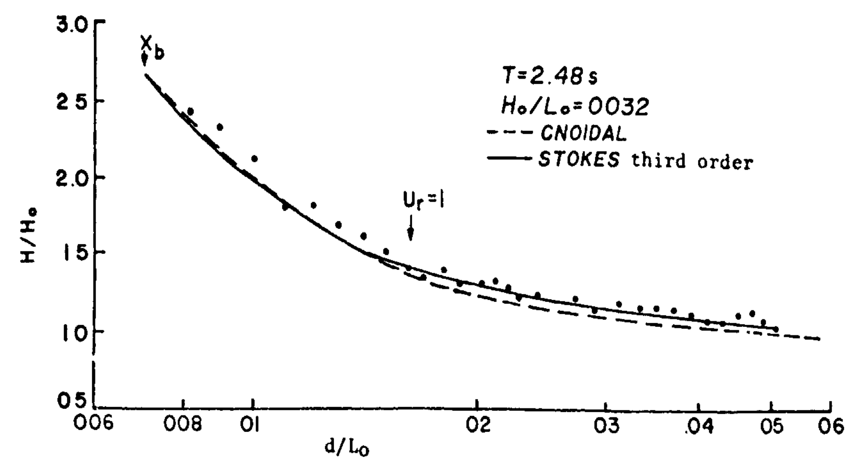


Figure 6. Comparison between Stokesian third order and cnoidal shoaling coefficient with experiments (from Flick, 1978).

Interestingly, the use of the linear wave theory to evaluate the value of the shoaling coefficient extends much beyond the formal validity of this infinitesimal wave theory. Similarly, the value of the shoaling coefficient given by the Stokesian wave theory extends into the area where the cnoidal theory fits best. This is due to the fact that, the shoaling coefficient being the ratio of wave height H/H_0 only, the increase in free-surface elevation under the crest is partly balanced by the increase of free-surface elevation under the wave trough. However, that the linear wave theory applies for the shoaling coefficient does not mean that all wave characteristics (wavelength, velocity components, pressure, acceleration) follow the same principle; after the local wave height is obtained, all other wave characteristics are determined by the appropriate theory.

3. Comparison Between Theory and Experiment.

A relatively large number of experiments have attempted to verify shoaling laws; all have been conducted in laboratory wave flumes with waves generated by wave paddle. Most of these experiments suffer lack of accuracy because they were either done at too small a scale and were subsequently subjected to significant scale effects such as large viscous damping experiments (Iversen, 1951), or the wave paddle generated not only monochromatic waves but harmonic components (solitons) which introduced significant error and scattering (Eagleson, 1956; Iwagaki, 1968).

There is actually considerable controversy whether waves of steady-state profile exist, as demonstrated by Dubreuil-Jacotin (1934). Theorists Benjamin and Feir (1967) and experimentalist Galvin (1970) postulate that the disintegration of finite amplitude monochromatic wave occurs in deep water even on horizontal bottom. There are as many theoreticians who assume that a steady-state profile does exist as there are experimentalists who do not notice the "creations" of solitons.

Use of a formulation developed by Mei and Le Mehaute (1966), Peregrine (1967), and Madsen and Mei (1969) indicates that for a sufficiently abrupt change in water depth, both a solitary wave and a cnoidal wave disintegrate into multiple crests. These results have been obtained numerically and verified experimentally. However, over a relatively gentle beach, the wave period remains constant between deep water and shallow water and no disintegration takes place. Disintegration takes place when the wave arrives on a reef. It seems natural to assume that the difference between these two observations is due to the difference in bottom slope. Benjamin and Feir (1967) show that waves are unstable if $kd > 1.4$; however, experiments by Flick (1978) indicate that kd can be much larger without evidence of wave disintegration or spectral smearing.

It is commonly accepted that a monochromatic wave arriving on a rapid change of depth (in diffraction zone) gives rise to at least a doubling of crests. Such phenomenon is due to the nonlinear convective effects. Iwagaki and Sagai (1971) also investigated the deformation of long waves over a gentle slope using the nonlinear long wave theory and power series expansions. They found the shoaling coefficient to be a function of beach slope when $S \gtrsim .01$. The steeper the slope the smaller the shoaling coefficient, a fact which can be attributed to partial reflection. In fact, due to friction effect, the ratio H/H_o for a given value of d/L_o decreases instead of increases (Sawaragi, Iwata, and Masayashi, 1976).

The first reliable experiments were conducted by Brink-Kjaer and Jonsson (1973) and Flick (1978). Figures 7 and 8 show results for different values of H_o/L_o. Flick separates the first, second, and third harmonics from his wave data and is subsequently able to give a reliable experimental shoaling coefficient. Flick compares his results with Le Mehaute and Webb (1964) (third-order Stokesian) and also with the cnoidal solution of Svendsen and Brink-Kjaer (1972) in shallow water (see Fig. 6).

The shoaling coefficient of a hyperbolic wave is also fairly well verified by Iwagaki (1968) who gives results very close to the two mentioned above.

Svendsen and Hansen (1977) compared the shoaling of cnoidal wave with a set of careful experiments and claimed that other experimenters (Wiegel, 1950, Iversen, 1951; Eagleson, 1956) carried out their experiments on too steep a slope for the shoaling theory to be valid. Furthermore, they calculate the damping due to viscous friction, obviously important on a gentle slope. Svendsen and Hansen concluded that if the wave height at depth $d/L_o = 0.10$ is matched between cnoidal and linear, rather than the energy flux, the cnoidal theory predicts the shoaling quite well, even close to breaking with small deepwater wave steepness $H_o/L_o < 3$ to 4 percent but not beyond. Consistently, with all theories, the wave just before breaking suddenly peaks up very rapidly (Le Mehaute, 1971). In this range of values, all shoaling theories (third Stokes, cnoidal and hyperbolic) tend to slightly underestimate the value of the shoaling coefficient. Subsequently, the calculated breaking wave height tends to be underestimated. The linear wave theory underestimates the breaking wave height most significantly, sometimes by a factor of almost 2 (Fig. 9).

It is pertinent to remember that (a) the shoaling coefficient given by the linear theory is valid beyond the limit generally considered as valid for a linear theory, and (b) the shoaling coefficient given by third-order Stokesian wave is fairly well verified experimentally and actually very close to the value given for the cnoidal wave, even though, as in the case of the linear wave, free-surface profile, pressure, velocity, and acceleration could be significantly different.

In general, the linear theory can be applied throughout from deep water to shallow water and then the linear breaking wave height is multiplied by a coefficient function of the beach slope (Koh and Le Mehaute, 1966). After the wave height, H, is determined as a function of the deepwater wave height, H_o, and wave period, T (or deepwater wavelength L_o), all other shallow-water characteristics (free-surface profile,

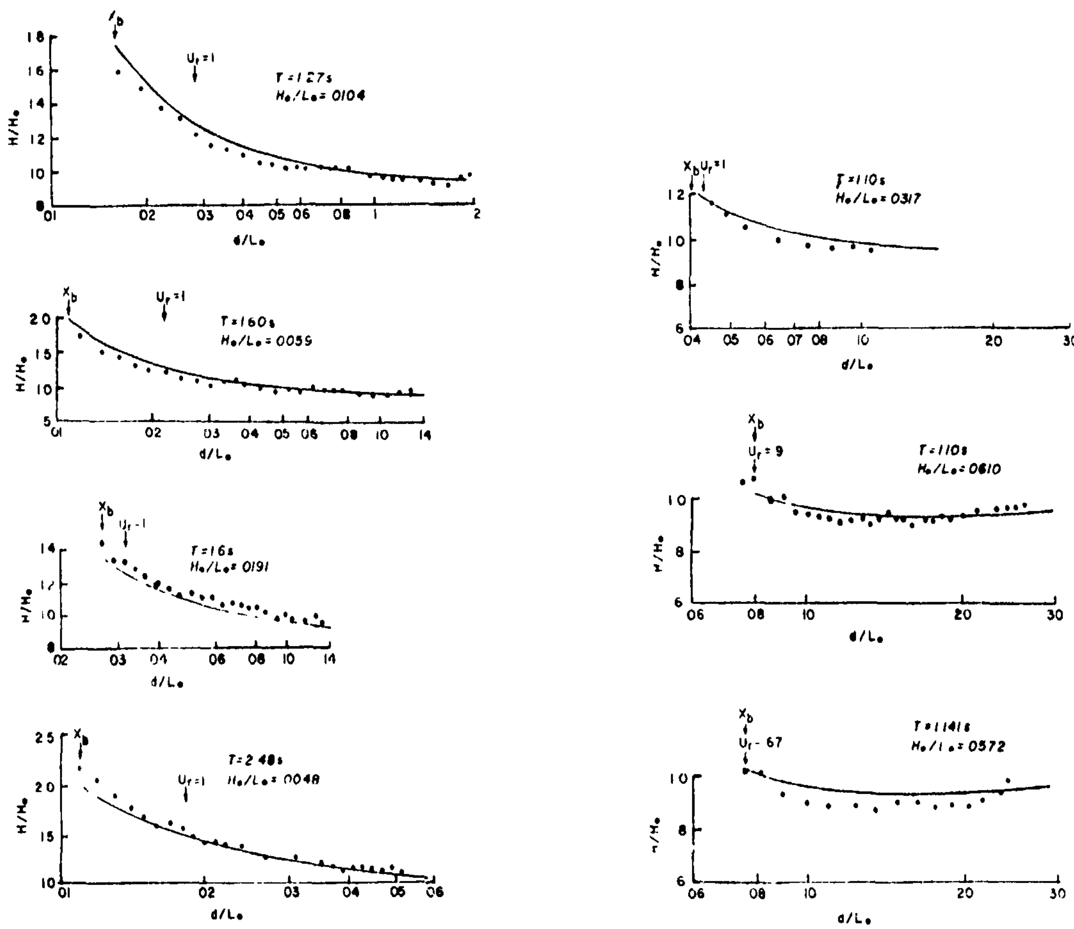
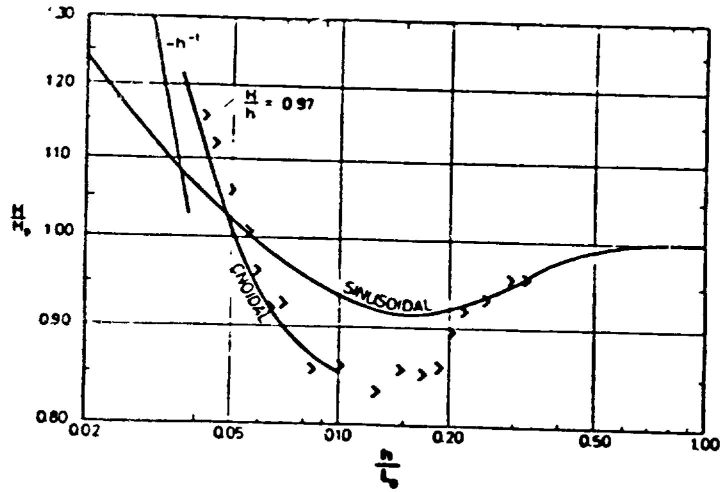
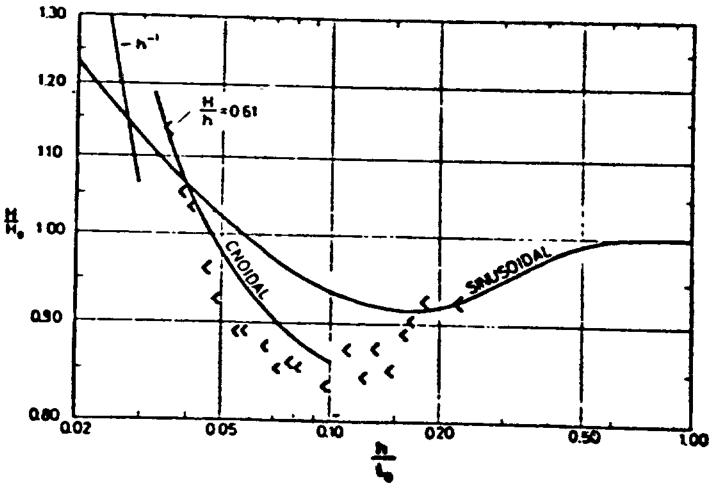


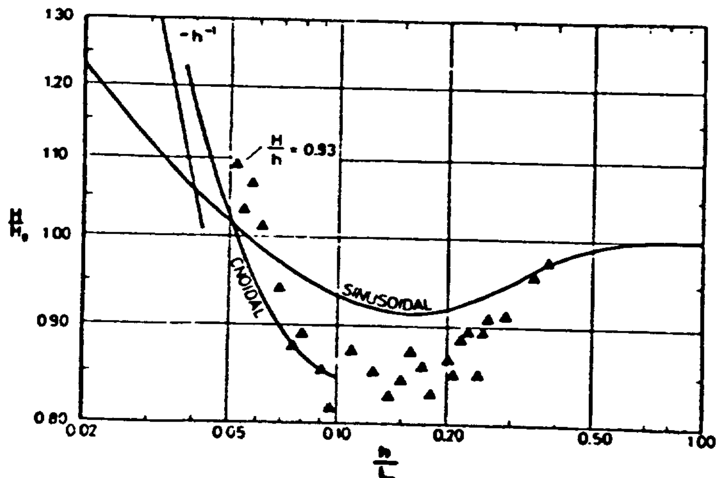
Figure 7. Comparison of experimental shoaling coefficients with Stokes third order (x_b is the breaking location) (from Flick, 1978).



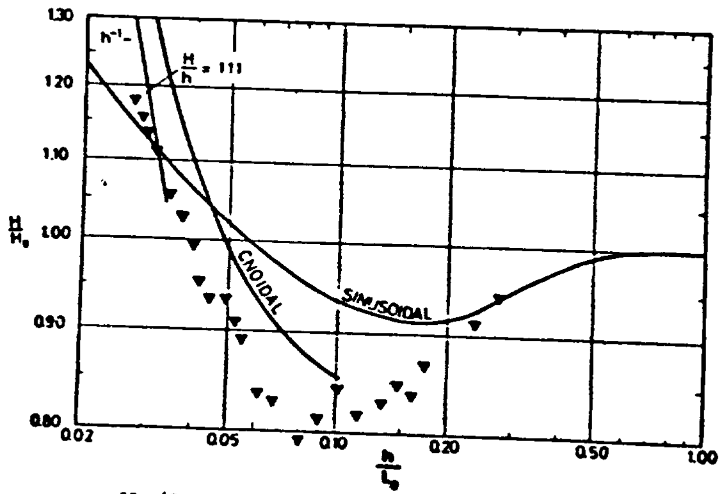
H_o/L_o = 0.0358; beach slope 0.072.



H_o/L_o = 0.0183; beach slope 0.072.



H_o/L_o = 0.0443; beach slope 0.072.



H_o/L_o = 0.0254; beach slope 0.054.

Figure 8. Experimental data for the shoaling coefficient K_s (from Brink-Kjaer and Jonsson, 1973).

particle velocity, acceleration, and pressure) follow by application of one of the classical wave theories within the accuracy which is determined for the chosen theory.

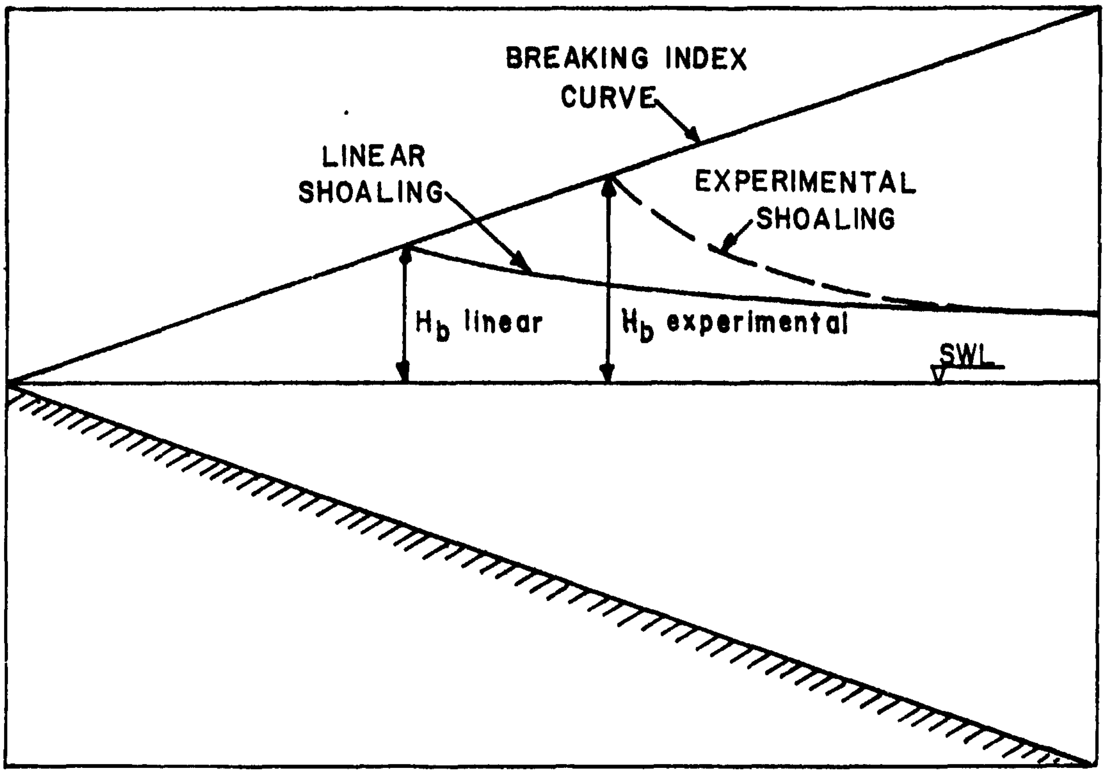


Figure 9. The linear theory underestimates the breaking wave height.

4. Nonlinear Wave Refraction.

It has been shown previously how to determine the shoaling coefficient, $K_s = H/H_0$, when the wave arrives perpendicular to the bottom contour. This discussion deals with the refraction coefficient, K_r. Refraction occurs when a wave arrives at an angle α with bottom contours; then $H/H_o = K_s K_r$. For a straight parallel contour Snell's law becomes

$$\frac{L}{\sin \alpha} = \text{constant} \tag{6}$$

which applies whether the wavelength, L, is expressed by a linear theory or not.

Then

$$K_r = \left(\frac{b}{b_o}\right)^{1/2} = \left(\frac{\cos \alpha}{\cos \alpha_o}\right)^{1/2} \tag{7}$$

which also applies for nonlinear as well as for linear theory. The subscript o refers to deepwater wave characteristics.

In many cases, the refraction method provides a reasonably accurate measure of the changes waves undergo on approaching a coast. However, if the angle of a wave ray with the bottom contour is large (i.e., larger than 70°), minor error in the value of the incident angle leads to a large error in direction angle α in shallow water. Also, accuracy as far as height changes are concerned cannot be expected where bottom slopes are steeper than 1/10. No strict limit has been set, but the accuracy of wave heights derived from orthogonals that bend sharply is questionable. In short, refraction coefficients which are quite different from unity, such as $K_r < 0.5$ and $K_r > 1.5$, must be doubted (Whalin, 1971).

Nonlinear effects, having an effect on wavelength, phase and group velocity and energy flux, subsequently have an effect on wave refraction. This problem has been examined by Chu (1975) who used a mix of three theories, i.e., the first-order cnoidal theory of Korteweg and DeVries (1895), the second-order hyperbolic wave of Iwagaki (1968), and the Stokes third-order wave as given by Le Mehaute and Webb (1964), which led to some inconsistencies in approximations. Skovgaard and Petersen (1977) used instead the first-order cnoidal theory of Svendsen (1974) and the stream function wave theories of Dean (1970).

Theoretically, it is possible to express phase velocities as a function of the relative wave heights from nonlinear wave theories. For example, the deepwater wavelength at a third order Stokesian approximation and the breaking wavelength by a cnoidal or hyperbolic wave theory can be conveniently expressed. However, it is interesting that due to deformation of wave profile on a sloped bottom, the simple linear theory has been verified (experimentally) quite well (Ippen, 1966). Wavelengths given by linear and cnoidal theories are compared in Figure 10. Although the cnoidal theory predicts wave height well up to breaking, it overpredicts wavelengths significantly. Cnoidal theory, in fact, predicts an increase in wavelength for a decrease in depth when the relative height, H/d, is sufficiently large. This increase is not reflected by known data (Ippen, 1966) which are fitted quite well by linear theory (Fig. 11).

III. BREAKING WAVE CHARACTERISTICS ON A SLOPED PLANE BEACH

1. Review of Previous Work.

The determination of longshore currents and sediment transport depends crucially on the characteristics of the breaking wave field. The wave energy transport, or energy flux, is of particular importance such that accurate determination of wave height, wavelength, depth at breaking, and breaking wave angle becomes essential.

This section deals with the practical aspects of determining the breaking wave characteristics when certain deepwater characteristics are given. The objective is to derive and present results

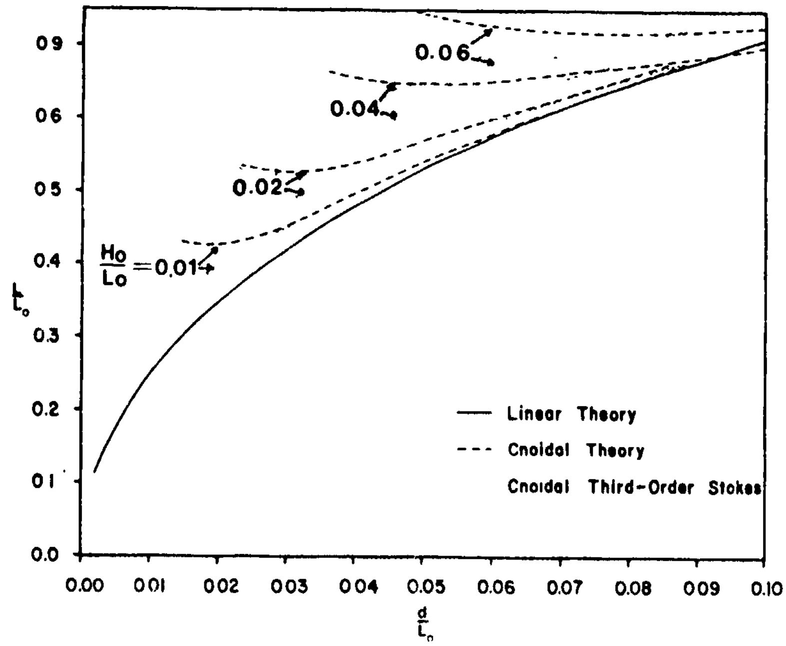


Figure 10. Wavelength as a function of depth predicted by different wave theories.

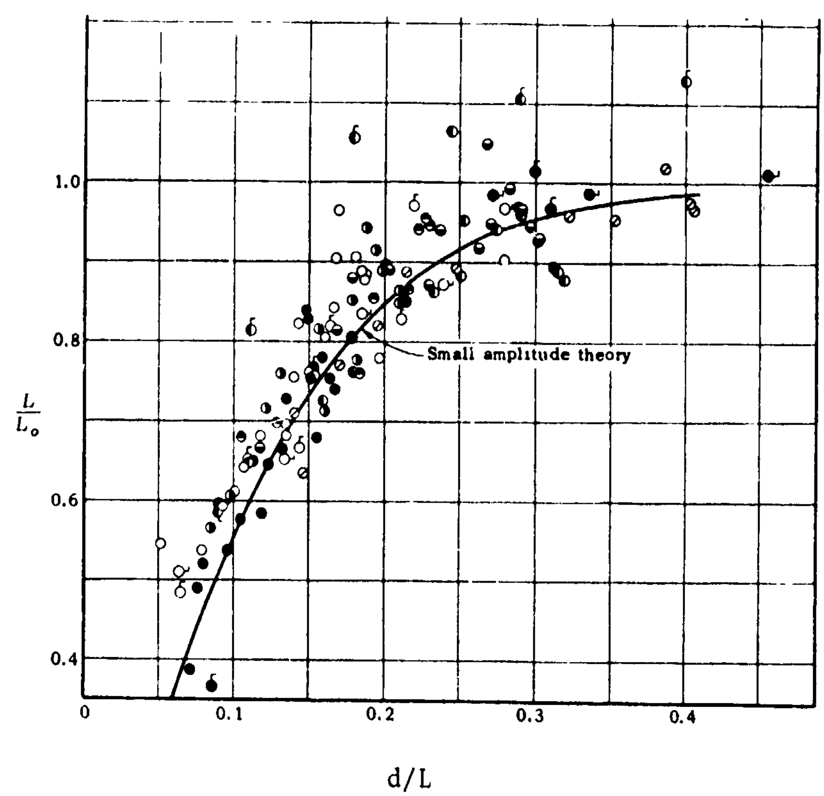


Figure 11. Transformation of wavelengths (from Ippen, 1966; used with the permission of McGraw-Hill Book Company).

consistent with present knowledge and in a readily usable form.

The general problem would require the determination of the shoaling and refraction of a multidirectional wave spectrum from deep water over a randomly varying bottom topography until breaking occurs. Although such an analysis is possible, it is much too complicated and would have to be dealt with on a case by case basis using either manual or computer methods.

A significant and useful simplification is achieved by assuming the bottom to be a uniformly sloped plane. This allows bottom variations to be described by a single parameter, i.e., the bottom slope S. The refraction process is then described globally by Snell's law. This discussion deals only with monochromatic waves under the usual assumption that the wave period remains constant, and that friction and reflection are ignored.

To obtain accurate predictions it is necessary to have a wave theory which is applicable up to the point of breaking. A lack of knowledge of the actual breaking process requires the use of an empirical breaking criterion to determine the point of breaking. A study by Le Mehaute and Koh (1967) evaluated the Stokes first-, third-, and fifth-order theories and compared the Miche (1944) breaking criterion with an empirically derived equation. One of these equations was derived by fitting a number of experimental data points covering the range $0.02 < S < 0.2$ and $0.002 < H_o/L_o < 0.09$. This equation explicitly accounts for beach slope and is

$$\frac{H_b}{H_o} = 0.76\ S^{1/7}\left(\frac{H_o}{L_o}\right)^{-1/4} \tag{8}$$

Since equation (8) is based on observed data it takes into account nonlinear effects such as wave height peak-up just before breaking. In applying this equation to waves arriving at an angle to the shore, Le Mehaute and Koh (1967) corrected the bottom slope for the angle of incidence; however, they neglected to replace the deepwater wave height with its unrefracted value.

Subsequently, a new and easier approach to compute cnoidal waves was presented by Svendsen (1974). Brink-Kjaer and Jonsson (1973) showed that near breaking the water depth is usually so shallow that cnoidal theory applies. Indeed, it has been found that wave height is described well by cnoidal theory in the area close to and before breaking.

In a recent report, Ostendorf and Madsen (1979) propose to use cnoidal and linear Stokes wave theories in their respective areas of applicability. A transition between the two theories which assumes continuous variation of energy flux and phase velocities is also presented. Ostendorf and Madsen further suggest the use of an empirical breaking criterion which is sensitive to bottom slope and depth-varying wave parameters, i.e.,

$$\frac{H_b}{L_b} = 0.14 \tanh \{(0.8 + 5S)\ 2\pi d_b/L_b\} \qquad S < 0.1$$

$$\frac{H_b}{L_b} = 0.14 \tanh \{(0.13)\ 2\pi d_b/L_b\} \qquad S > 0.1 \tag{9}$$

To obtain the breaking wave characteristics, the two offshore parameters ($\sin \alpha/C^*$, C_4) must be known where

α = angle of incidence

C = wave phase speed $C^* = C/(gT)$

g = gravity acceleration

T = wave period

$C_4 = T \sqrt{g}\ \{(\frac{H}{h})^2\ n \sin \alpha)\}^{-1/4}$

d = stillwater depth

$n = \frac{1}{2}\ (1 + \frac{2kd}{\sinh 2kd})$

$k = 2\pi/L$

L = wavelength

In the deepwater limit this implies that wave height, H_o, angle of incidence, α_o, and wave period must be known independently. The solution requires an iteration process and nomographs are presented to facilitate the operations.

The method for determining breaking wave characteristics suggested by Ostendorf and Madsen (1979) has been compared with experimental data (Kamphuis, 1963), and it was found that the predicted breaking wave angle is too large, especially for smaller wave steepnesses (see Table 1). This is easily explained when considering the plot of wavelength transformation shown in Figure 10. Although cnoidal theory predicts wave height well up to breaking, it overpredicts wavelengths significantly. Cnoidal theory, in fact, predicts an increase in wavelength and therefore, also in wave angle when H/d is sufficiently large. This increase is, as previously mentioned, not reflected by known data (Ippen, 1966; Fig. 11), which are fitted quite well by linear theory. As another consequence, the wave breaking criterion, again, a result which is difficult to defend.

Dean (1974) determined wave breaking angles using his stream function theory, but with a slope-independent semiempirical breaking criterion. A comparison of his results with the experimental observations of Kamphuis (1963) is also presented in Table 1. The predictions are consistently too high, especially for smaller wave steepness, where predicted and observed α_b differ by a factor of approximately 2.

Table 1. Comparison of measured and predicted breaking wave angles.[1]

$\frac{H_o}{o}$		0.0175			0.04		0.053		0.062	
α_o		20^o	40^o	60^o	20^o	40^o	20^o	40^o	20^o	40^o
α_b	Kamphuis (1963)	5^o	9^o	12^o	8^o	16^o	10^o	20^o	11^o	22^o
α_b	Ostendorf and Madsen (1979)	10^o	18.6^o	24.3^o	12.8^o	24.3^o	13.9^o	26.5^o	14.5^o	28.8^o
α_b	Dean (1974)	9.5^o	19^o	22^o	13^o	25^o	14^o	27^o	14.5^o	28^o

[1]Beach slope, S = .1

2. Solution Approach.

To obtain reliable prediction of breaking wave characteristics, this study proposes to use cnoidal theory to describe the transformation of wave height while wavelength will be transformed using either linear wave theory or third-order Stokes theory. The cnoidal wavelength is then considered as an auxiliary parameter which cannot be identified as the physical wavelength. Linear wave theory is simpler to use; however, to retain some nonlinear effect in the transformation of wavelength the third-order Stokes theory is also included. The wavelength computed using the cnoidal third-order Stokes theory is shown in Figure 11.

Due to the large wave heights near breaking, a higher order approximation to cnoidal waves given by Iwagaki (1968) was considered. A more detailed discussion of this "hyperbolic" wave theory is given in the Appendix. Note that this higher order theory suffers from the same problem of inhomogeneous convergence as, for example, plagues fifth-order Stokes waves (Le Mehaute and Koh, 1966), and therefore gives poorer results than the first-order cnoidal theory near breaking.

The computation and shoaling of cnoidal waves have been given by Svendsen (1974) and Svendsen and Brink-Kjaer (1972). It is convenient to define the parameter

$$U = \frac{HL_c^2}{d^3} = \frac{16}{3} mK^2(m) \tag{10}$$

where

H = wave height

L_c = cnoidal wavelength parameter

d = stillwater depth

K(m) = complete elliptic integral of the first kind

The dispersion relationship may be written as

$$\frac{L_c}{L_o} = \sqrt{\frac{d}{L_o} 2\pi \left(1 + \frac{H}{d} A\right)} \qquad (11)$$

The deepwater wavelength $L_o = \frac{g}{2\pi} T^2$ and $A = A(m) \equiv \frac{2}{m} - 1 - \frac{3E}{mK}$

The complete elliptic integral of the second kind is designated E.

Invoking energy flux conservation between wave rays and using linear theory in deep water, the wave height transformation is given as

$$\frac{H}{H_o} = \left(\frac{1}{16}\right)^{2/3} \left(\frac{H_o}{L_o}\right)^{1/3} \left(\frac{d}{L_o}\right)^{-1} f_H \qquad (12)$$

In equation (12) $f_H = f_H(U) \equiv U^{-1/3} \cdot B^{-2/3}$

and

$$B = B(m) \; \frac{1}{m^2} \left[\frac{1}{3} \left(3m^2 - 5m + 2 + (4m - 2) \frac{E}{K}\right) - \left(1 - m - \frac{E}{K}\right)^2\right] \qquad (13)$$

Equations (12) and (13) define the shoaling of cnoidal waves and are used to determine the shoaling coefficient.

To compute the wavelength and refraction the Stokes wave theory is used. Linear waves are described by the dispersion relationship

$$\frac{L_1}{L_o} = \tanh \frac{2\pi d}{L_1} \qquad (14)$$

For the third-order Stokes approximation from Le Mehaute and Webb (1964)

$$\frac{L_3}{T^2} = \frac{g}{2\pi} \tanh \frac{2\pi d}{L_3} \left(1 + C_1 \lambda_3^2\right) \qquad (15)$$

$$\frac{L_{3o}}{T^2} = \frac{g}{2\pi} \left(1 + \lambda_{3o}^2\right) \qquad (16)$$

where

$$\frac{3}{8} \lambda_{3o}^3 + \lambda_{3o} = \frac{\pi H_o}{L_{3o}} \qquad (17)$$

$$B_{33}\,\lambda_3^3 + \lambda_3 = \frac{\pi H}{L_3} \tag{18}$$

$$C_1 = \frac{8\cosh^4 k_3 d - 8\cosh^2 k_3 d + 9}{8\sinh^4 k_3 d} \tag{19}$$

$$B_{33} = \frac{3(8\cosh^6 k_3 d + 1)}{64\sinh^6 k_3 d} \tag{20}$$

and

$$k_3 = \frac{2\pi}{L_3} \tag{21}$$

Refraction is described by Snell's law

$$\frac{L}{L_o} = \frac{\sin\alpha}{\sin\alpha_o} \tag{22}$$

and the refraction coefficient

$$K_R = \sqrt{\frac{\cos\alpha_o}{\cos\alpha}} \tag{23}$$

where α = angle of wave with shoreline. Also, when refraction is included the deepwater wave height in the above expressions should be replaced by its unrefracted value.

Finally, the point of breaking is determined by a modified form of the empirical breaking criterion (eq. 8). To adapt the formula to waves approaching at an oblique angle, the bottom slope and deepwater wave height are replaced by S $\cos\alpha_b$ and H_o $(\cos\alpha_o/\cos\alpha_b)^{1/2}$, respectively. The breaking criterion is then obtained by

$$\frac{H_b}{H_o'} = 0.76\ S^{1/7} \cos^{1/7} \alpha_b \left(\frac{H_o'}{L_o}\right)^{-1/4} \tag{24}$$

or

$$\frac{H_b}{H_o} = K_r\ 0.76 \cos^{1/7} \alpha_b \left(\frac{H_o}{L_o S^{4/7}}\right)^{-1/4} K_r^{-1/4} \tag{25}$$

Rewriting equation (25) using $H_b/H_o = K_s K_r$, where K_s is the shoaling coefficient yields

$$K_{sb} = 0.76 \left(\frac{H_o}{L_o S^{4/7}}\right)^{-1/4} \cos^{-1/8} \alpha_o \cos^{15/16} \alpha_b \tag{26}$$

Equation (26) gives the form of the breaking criterion used in this study.

An explicit solution for the breaking wave characteristics cannot be obtained from equations (11) to (23) and equation (26). A numerical solution is required and it becomes important to reduce the number of independent parameters as much as possible. By straightforward manipulation of the equations, only three independent parameters need to be known: deepwater wave steepness, H_o/L_o; beach slope, S; and deepwater incident wave angle, α_o.

A computer algorithm is constructed to solve the problem. The basic process consists of the following steps:

(1) Give values for $\frac{H_o}{L_o}$, S, α_o

(2) Assume $K_s=K_r=A=1$ and $\left\{\begin{array}{c}\frac{L_{1b}}{L_{1o}} \\ \text{or } \frac{L_{3b}}{L_{3o}}\end{array}\right\} = 0.4$

(3) Guess a value for $\frac{H_b}{d_b}$

(4) Find

$$\frac{L_{cb}}{L_o} = \sqrt{\left(\frac{H_b}{d_b}\right)^{-1} K_s\, K_r\, \frac{H_o}{L_o}\, 2\pi \left(1 + \frac{H_b}{d_b} A\right)}$$

(5a) Determine $\frac{H_b}{L_{1b}} = K_s\, K_r \times \frac{H_o}{L_o} \left(\frac{L_{1b}}{L_{1o}}\right)^{-1}$

and $k_{1b}\, d_b = 2\pi \left(\frac{H_b}{d_b}\right)^{-1} \frac{H_b}{L_{1b}}$

(5b) Determine $\frac{H_b}{L_{3b}} = K_s\, K_r \times \frac{H_o}{L_o} \left(\frac{L_{3b}}{L_{3o}}\right)^{-1}$

and $k_{3b}\, d_b = 2\pi \left(\frac{H_b}{d_b}\right)^{-1} \frac{H_b}{L_{3b}}$

(6a) Find $\frac{L_{1b}}{L_{1o}} = \tanh k_{1b}\, d_b$

(6b) Find $\frac{L_{3b}}{L_{3o}} = \tanh k_{3b}\, d_b \left(\frac{1+C_1\lambda_3}{1+\lambda^2_{o3}}\right)$

(7) Find α_b from equation (22), using wavelength ratio from step (6), and find K_r from equation (16)

(8) Determine $U = \frac{H_b}{d_b} \times \left(\frac{L_c b}{L_o} \cdot \frac{L_o}{d_b}\right)^2$

(9) Find m and K(m) satisfying equation (10) using polynomial approximation of K given in Abramovitz and Stegun (1964)

(10) Determine E, A, B, f_H and then K_s from equation (12)

(11) Go back to step (4) until K_s remains constant

(12) Compare the obtained K_s value with value computed from equation (26). If different, go back to step (3)

3. Results.

The wave breaking angle, α_b, as computed with linear, cnoidal-linear, and cnoidal third-order Stokes approaches, is compared with the experimental data of Kamphuis (1963) as shown in Le Mehaute and Koh (1967) and Figure 12. These are apparently the only data on α_b and are obtained for a single bottom slope, S = 0.1. For H_o/L_o = 0.0175, linear theory gives the best fit to the data. On the other hand, cnoidal linear theory provides a better fit for the larger steepnesses, H_o/L_o = 0.053 and 0.062. Unfortunately, it is not possible to draw any definite conclusions from the limited data. However, it appears that linear theory provides the best estimate of wavelength irrespective of wave steepness, as found by Eagleson and Dean (Ippen, 1966). Also, for large wave steepness, cnoidal theory predicts the wave height quite accurately up to the point of breaking.

For a given beach slope the waves break at increasing relative depth ratios, d/L, as the deepwater steepness increases. For large enough H_o/L_o, the cnoidal theory is no longer valid since it predicts a nonphysical complex wave height (Svendsen, 1974). The critical deepwater wave steepness for which the cnoidal theory ceases to be valid has been determined for five different bottom slopes (Table 2). This critical value is only weakly sensitive to the magnitude of α_o. For H_o/L_o greater than the critical value, a different wave theory such as Dean's (1974) or third-order Stokes (Le Mehaute and Koh, 1967) must be used.

In Figures 13 to 17, the variation of breaking wave angles versus deepwater wave angle with H_o/L_o and S as parameters is depicted as computed using the cnoidal-linear theory. Similar results for linear theory are shown in Figure 18, when H_o/L_o and S are combined into a single parameter.

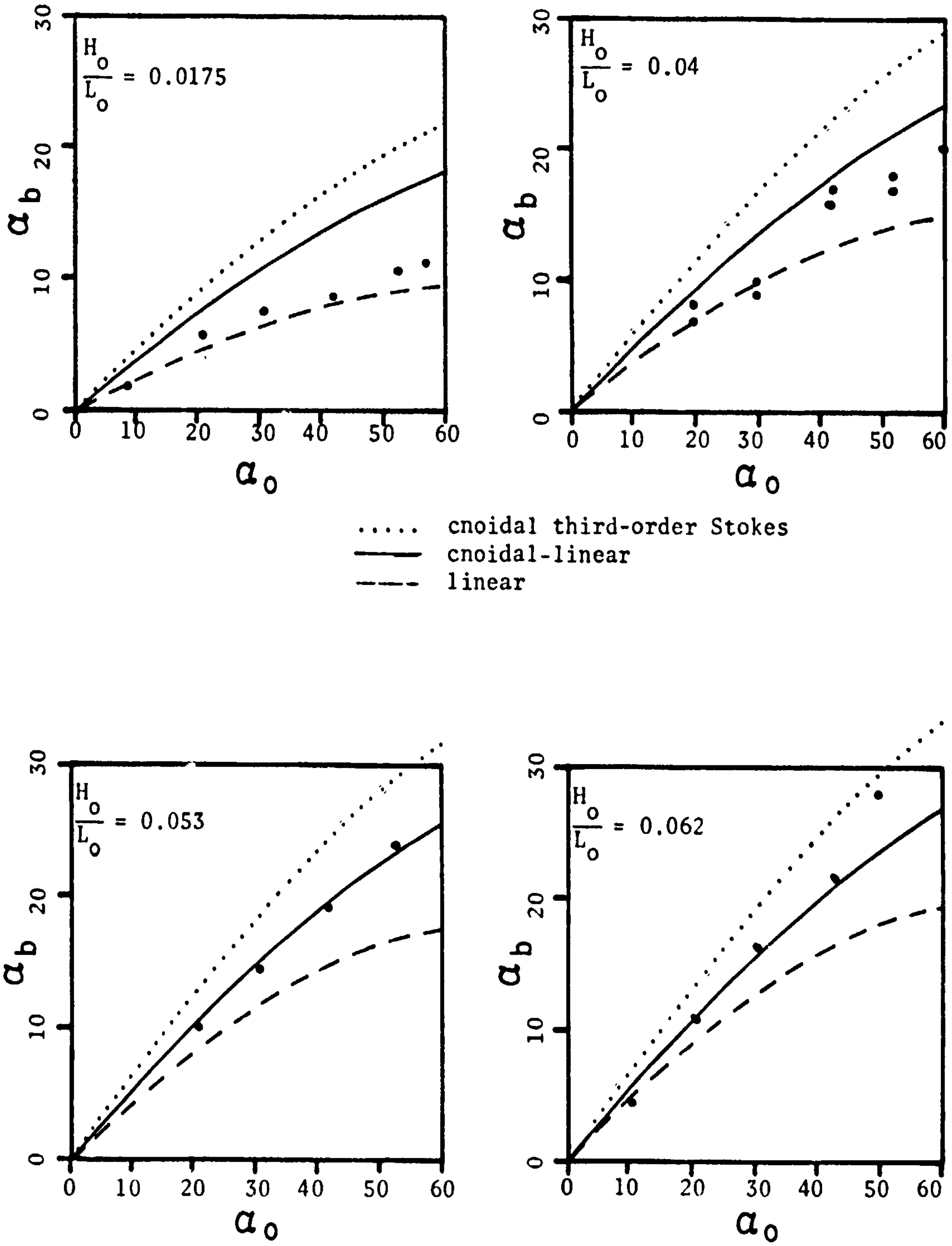


Figure 12. Comparison of experimental and theoretical wave breaking angles (bottom slope, S = 0.10).

Table 2. Limit for validity of cnoidal theory for normal incidence.

S	$\frac{H_o}{L_o}$	$\frac{H_b}{d_b}$
.01	.046	0.385
.02	.076	0.576
.03	.107	0.741
.04	.139	0.872
.05	.173	0.993

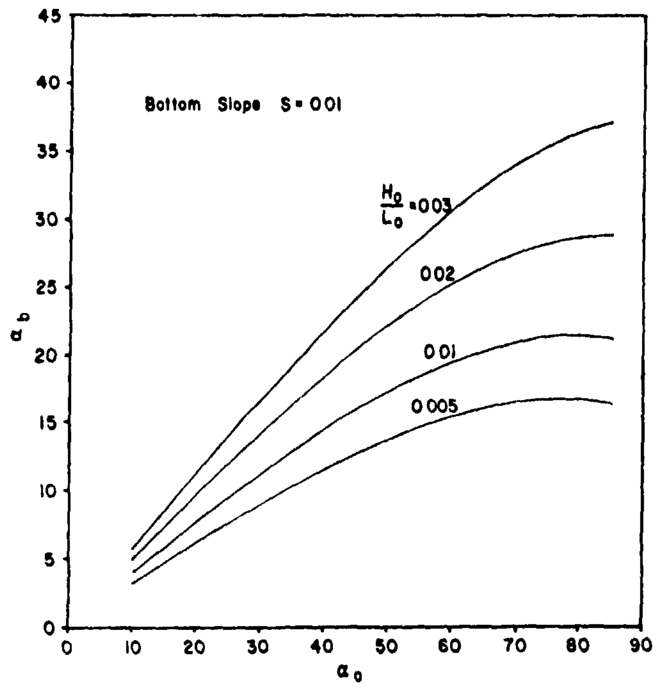


Figure 13. Wave breaking angle, α_b, as a function of deepwater wave angle, α_o, and deepwater steepness, H_o/L_o, on a bottom slope, S = 0.01.

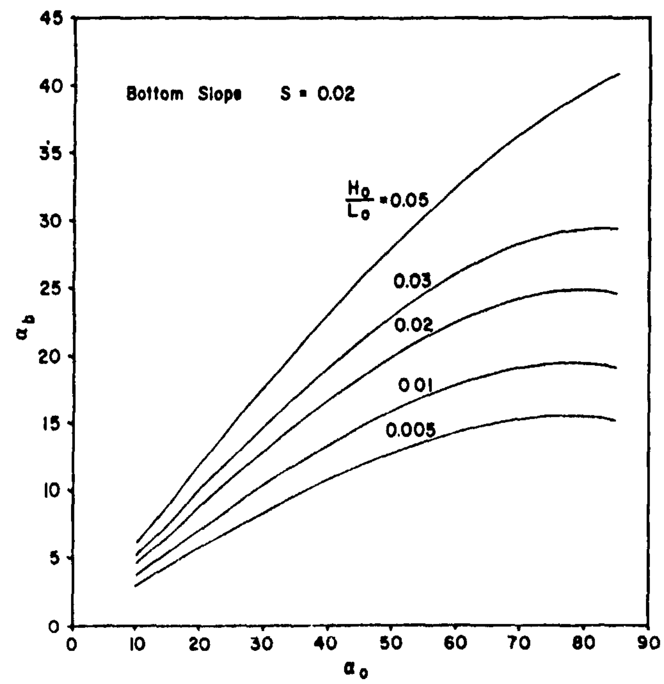


Figure 14. Wave breaking angle, α_b, as a function of deepwater wave angle, α_o, and deepwater steepness, H_o/L_o, on a bottom slope, S = 0.02.

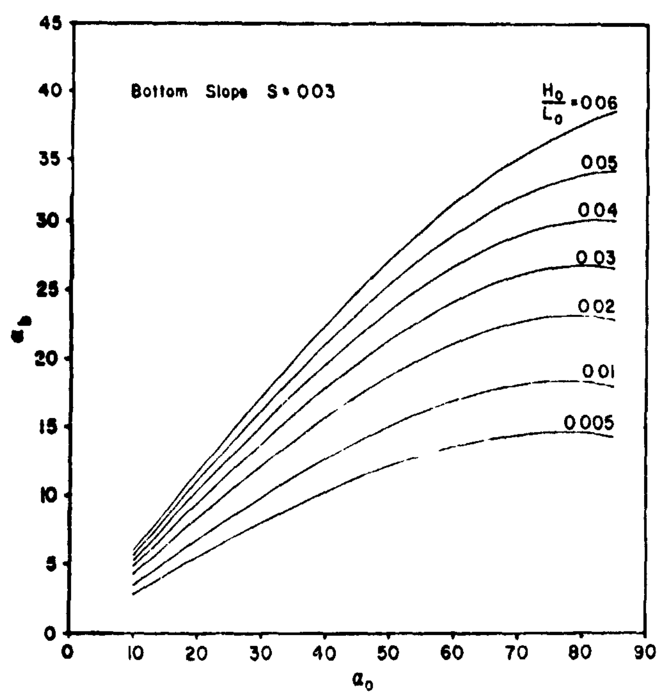


Figure 15. Wave breaking angle, α_b, as a function of deepwater wave angle, α_o, and deepwater steepness, H_o/L_o, on a bottom slope, S = 0.03.

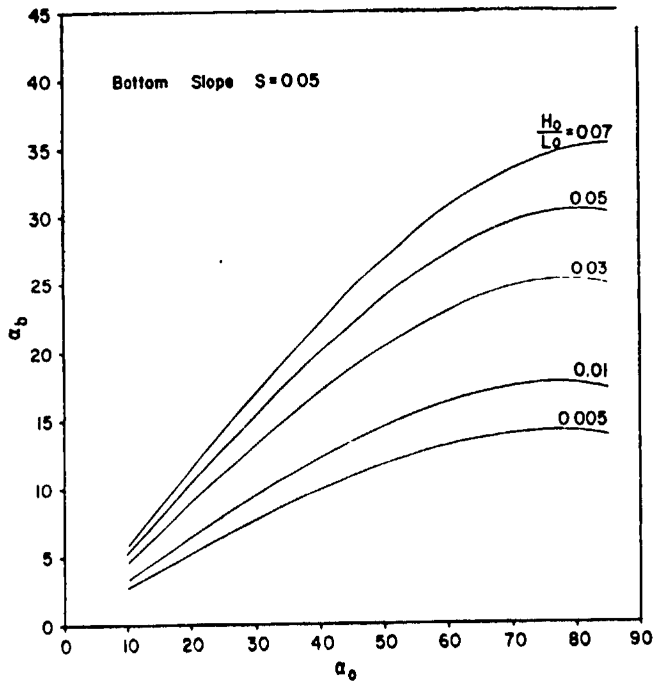


Figure 16. Wave breaking angle, α_b, as a function of deepwater wave angle, α_o, and deepwater steepness, H_o/L_o, on a bottom slope, S = 0.05.

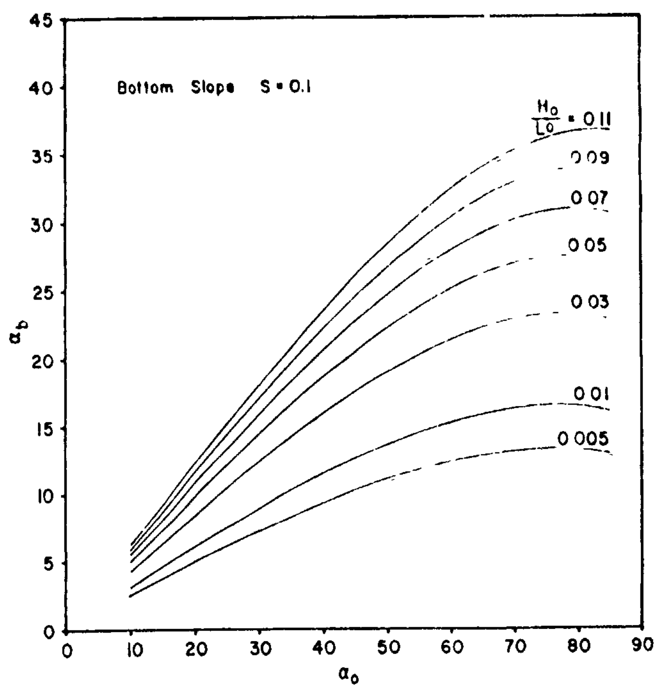


Figure 17. Wave breaking angle, α_b, as a function of deepwater wave angle, α_o, and deepwater steepness, H_o/L_o, on a bottom slope, S = 0.10.

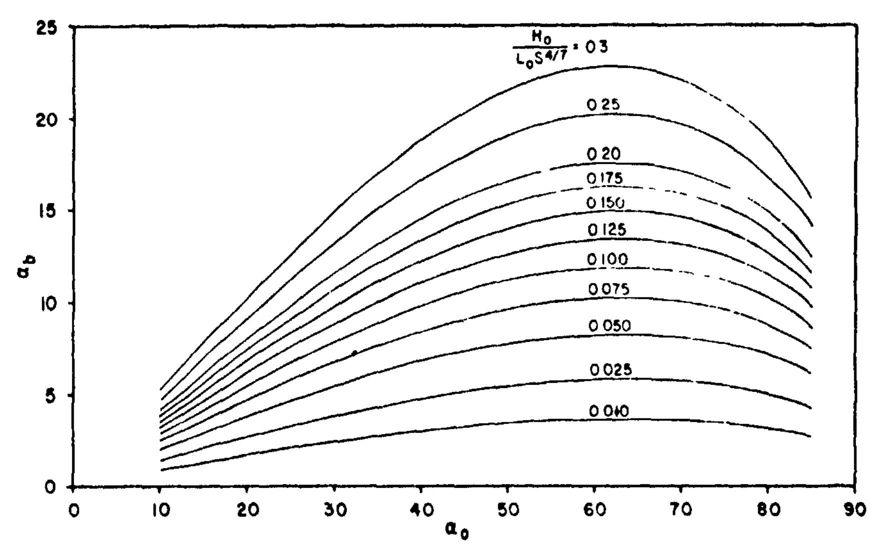


Figure 18. Prediction of wave breaking angles using linear theory.

The remaining breaking wave characteristics are easily determined as shown in the following example:

Given: $\frac{H_o}{L_o} = 0.02$ $S = 0.03$ and $\alpha_o = 40^o$

Find: α_b, $\frac{d_b}{L_o}$ and $\frac{H_b}{H_o}$

(a) Cnoidal-linear

From Figure 15 is found

$$\alpha_b = 15^o.9$$

Using Snell's law

$$\frac{L_b}{L_o} = \frac{\sin\alpha_b}{\sin\alpha_o} = 0.426$$

The wave height can be found from the breaking criterion

$$\frac{H_b}{H_o} = 0.76 \cos^{1/7} \alpha_b \left(\frac{H_o}{L_o S^{4/7}}\right)^{-1/4} K_r^{3/4} = 1.118$$

Finally, the depth of breaking is computed from the dispersion relationship, which is solved for

$$k_b\, d_b = \frac{1}{2} \ln \left| \frac{1 + \frac{L_b}{L_o}}{1 - \frac{L_b}{L_o}} \right| = 0.455$$

or

$$\frac{d_b}{L_b} = 0.0724 \text{ and } \frac{d_b}{L_o} = \frac{d_b}{L_b} \cdot \frac{L_b}{L_o} = 0.0308$$

(b) Linear theory

Determine $\dfrac{H_o}{L_o S^{4/7}} = 0.148$

From Figure 18

$$\alpha_b = 12.2^o$$

Snell's law $\dfrac{L_b}{L_o} = \dfrac{\sin\alpha_b}{\sin\alpha_o} = 0.329$

Wave height $\dfrac{H_b}{H_o} = 0.76 \cos^{1/7} \alpha_b \left(\dfrac{H_o}{L_o S^{4/7}}\right)^{-1/4} K_r^{3/4} = 1.115$

Depth at breaking

$$k_b\, d_b = \frac{1}{2} \ln \left| \frac{1 + \frac{L_b}{L_o}}{1 - \frac{L_b}{L_o}} \right| = 0.3417$$

or

$$\frac{d_b}{L_b} = .0544 \text{ and } \frac{d_b}{L_o} = \frac{d_b}{L_o}\frac{L_b}{L_o} = 0.0179$$

IV. SUMMARY AND CONCLUSIONS

It appears that no single wave theory accurately predicts the transformation of a wave from deep to shallow water. In shallow water, cnoidal theory successfully describes the shoaling of wave height but overestimates the wave celerity and wavelength. All previous studies emphasize an accurate determination of wave height, but little attention is paid to the wavelength.

When considering longshore currents and sediment transport the angle of wave breaking becomes a parameter of high importance and, therefore, also the wave refraction process which is intimately connected with wavelength.

In this study a "hybrid" wave theory is proposed, consisting of cnoidal wave height and linear wavelength transformation in the regime where cnoidal theory applies. In comparison with existing data this theory is found to predict wave height and wavelength better than previous theories, especially for larger wave steepnesses. More definite conclusions must await new experimental data. This basic information seems to be missing in the existing literature.

Nomographs are presented for determining the breaking wave characteristics for given deepwater characteristics and bottom slope for a plane beach.

The remaining difficulties encountered in the shoaling and refraction of a monochromatic wave are further amplified when dealing with a more realistic directional spectrum of waves. What are the equivalent monochromatic breaking wave characteristics that produce the same sediment transport or longshore current? This question seems to be one of the more important to face in the future research.

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APPENDIX

Hyperbolic Wave Shoaling

It is well known that near breaking surface gravity water waves are highly nonlinear and therefore not well characterized by linear theory. For prediction of breaking wave characteristics it is necessary to use higher order wave theories and, in particular, a higher order cnoidal wave solution could appear to be attractive because of the improved fit to data found when using first-order cnoidal theory. Iwagaki (1968) derived a practical asymptotic solution to second-order cnoidal waves called "hyperbolic waves." This approximation takes $m = 1$ and $E(m) = 1$ for $K(m) > 3$ where $K(m)$ and $E(m)$ are the complete elliptical integrals of the first and second kind.

If the wave height transformation is written in the usual manner as a product between a shoaling and refraction coefficient

$$H = H_o K_s \cdot K_r \tag{A-1}$$

then Iwagaki derived

$$K_s = \frac{3}{16}\left(\frac{1}{4}\right)^{1/3}\left(\frac{d}{L_o}\right)^{-1}\left(\frac{H_o}{L}\right)^{1/3}\left\{1 + \pi^2\frac{H_o'}{L_o}^{2}\right\}$$
$$\left\{1 - \frac{1}{K}\frac{H}{d} + \frac{1}{12}\frac{1}{K}\left(\frac{H}{d}\right)^2\right\}^{-1/3}\left\{1 - a\left(\frac{H}{d}\right)^n\right\}^{2m/3}$$
$$\left\{1 - \frac{3}{2}\frac{1}{K} + \frac{H}{d_t}\left(\frac{2}{5} - \frac{5}{2}\frac{1}{K} + \frac{3}{K^2}\right) + \left(\frac{H}{d_t}\right)^2\left(-\frac{31}{112} - \frac{29}{160}\frac{1}{K} - \frac{13}{4}\frac{1}{K^2}\right)\right\}^{-2/3} \tag{A-2}$$

where

d = water depth

$L_o = \frac{gT^2}{2\pi}(1 + \lambda_o^2)$ = deepwater wavelength (third-order Stokes wave)

H_o' = unrefracted deepwater wave height

K = complete elliptic integral

H = wave height

$d_t = d\{1 - \frac{1}{K}\frac{H}{d} + \frac{1}{12}\frac{1}{K}(\frac{H}{2})^2\}$ = water depth under trough

The parameter λ_o is determined from

$$\lambda_o = \pi \frac{H_o}{L_o} \cdot \frac{1}{\frac{3}{8}\lambda_o^2 + 1} \tag{A-3}$$

and Iwagaki found by empirical curve fitting

$$a = 1.3,\ n = 2 \text{ and } m = \frac{1}{2} \text{ for } \frac{H}{d} \leq 0.55$$

$$a = 0.54,\ n = \frac{3}{2} \text{ and } m = 1 \text{ for } \frac{H}{d} > 0.55$$

Such that the elliptic integral, K, could be approximated by

$$\frac{K}{T\sqrt{g/d}} = \frac{\sqrt{3}}{4} \left(\frac{H}{d}\right)^{1/2} \left\{1 - a \left(\frac{H}{d}\right)^n\right\}^m \tag{A-4}$$

Finally, in hyperbolic waves the wavelength is given by

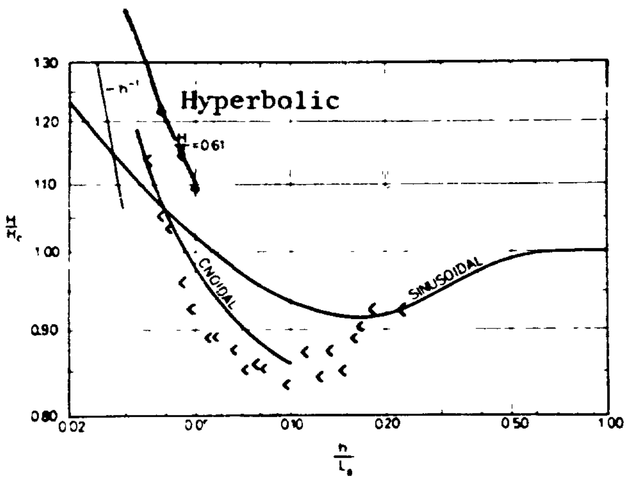
$$\frac{L}{L_o} = \sqrt{2}\pi \left(\frac{d}{L_o}\right)^{1/2} \left\{1 - \frac{\pi^2}{2} \left(\frac{H'_o}{L_o}\right)^2\right\} \left(1 - \frac{3}{2}\frac{1}{K}\frac{H}{d}\right)$$

$$\times \left\{1 - a \left(\frac{H}{d}\right)^n\right\}^m \left\{1 - \left(1 + \frac{1}{K}\frac{H}{d}\right) \frac{5}{8}\frac{H}{d}\right\}^{-1} \tag{A-5}$$

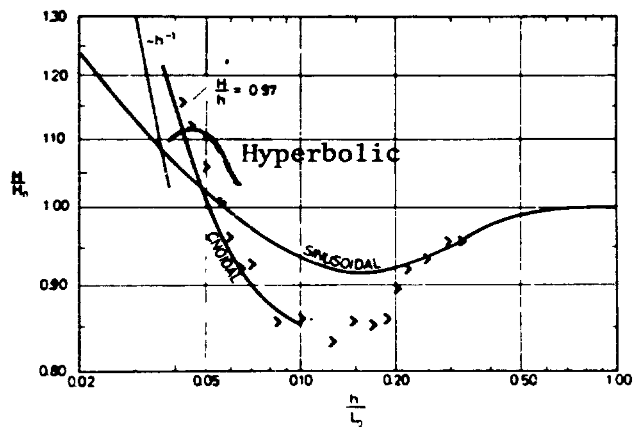
Again, it should be pointed out that equations (A-2) to (A-5) are only valid for $K \geq 3$.

To evaluate the hyperbolic wave theory the shoaling factor K_s is compared with data and first-order cnoidal theory (Brink-Kjaer and Jonsson, (1973). The results are shown in Figure A-1.

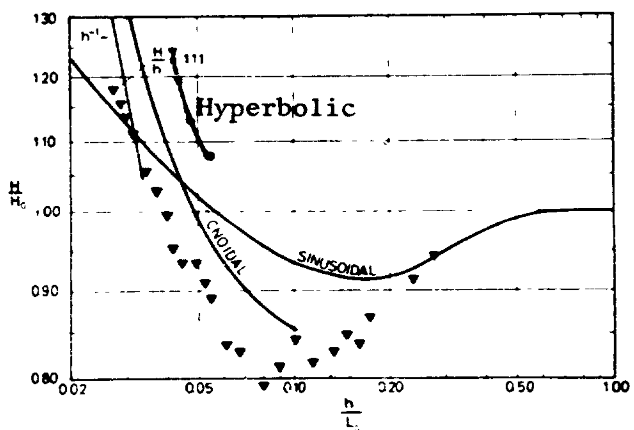
It is evident that the results given by the hyperbolic theory are worse than those obtained using linear cnoidal theory. For large wave steepness the hyperbolic theory exhibits a decrease in K_s near breaking similar to fifth-order Stokes waves which is also due to nonhomogeneous convergence of the perturbation series. Finally, the wavelength predicted by hyperbolic theory (not shown here) is quite close to the cnoidal wavelength which is in itself questionable as examined earlier. Further attempts at using hyperbolic wave theory to predict breaking wave characteristics were abandoned because of the above shortcomings.



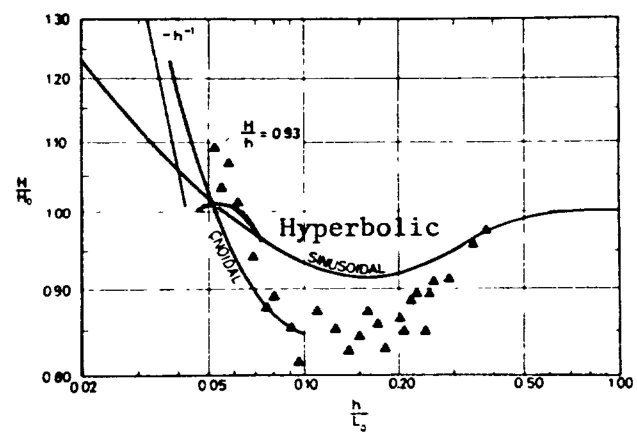
Run 4: H_0/L_0 = 0.0183. Beach slope 0.072



Run 3: H_0/L_0 = 0.0358. Beach slope 0.072



Run 9: H_0/L_0 = 0.0254. Beach slope 0.054



Run 2: H_0/L_0 = 0.0443. Beach slope 0.072.

Figure A-1. Comparison of hyperbolic wave theory with data and cnoidal theory (from Brink-Kjaer and Jonsson, 1973).

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